**On project design:**

First, I created the BaseOption class, which is used to hold the parameters of an option and is responsible for setting and reprieving these parameters. It has a data member called param in the form of a map<string, double>. I chose this container because it can be used to hold as many option parameters as necessary, thus adaptable for other option types. Derived classes can add parameters simply by adding new elements to the map. The map’s key is also useful for setting and retrieving specific parameter. Thus there is no need to create a getter and setter function for each parameter. The only parameter initialized in BaseOption’s constructor is risk-free interest rate, since this is a macro-economic factor independent of option characteristics. BaseOption is set as an abstract class since it needs payoff functions and pricing formulas to be useful.

The EuropeanOption class is derived from BaseOption and contains functions to calculate price, delta, and gamma. It also implements put-call parity and checks if the parity holds. It has no new data member, since relevant parameters can be entered to BaseOption’s param.

The Mesher.hpp file contains utility functions for creating vector instances. It also includes printMesh(), which helps print out a vector.

The BaseMatrix class is used to hold a matrix of option parameters. This matrix is in the form of map<string, vector<double>>. Each vector<double> is for one numeric parameter. This class is responsible for setting and reprieving these parameters. Users can enter parameters of different sizes (either a double or a vector<double>). The updateParam() function would find the size of the longest parameter, then extend all other parameters to that size. This size is the number of options to be priced as well as the size of the output price vector. The goal here is to make the process of inputting parameters more robust. Like BaseOption, BaseMatrix is an abstract class.

The EuropeanMatrix class is derived from BaseMatrix and is responsible for calling the proper pricing and sensitivities calculation functions from EuropeanOption. In addition, it can call the ddDelta and ddGamma global functions to calculate delta and gamma for multiple options using divided differences.

Lastly, the DividedDifferences.hpp contains the ddDelta and ddGamma template global functions. These are written as template global functions because the divided differences formulae apply to all options, not just European.

**Outputs:**

A.1.a and A.1.b:

Batch 1

Call price:

2.13337

Put price from parity:

5.84628

Put price:

5.84628

Call price from parity:

2.13337

Put-call parity holds

Batch 2

Call price:

7.96557

Put price from parity:

7.96557

Put price:

7.96557

Call price from parity:

7.96557

Put-call parity holds

Batch 3

Call price:

0.204058

Put price from parity:

4.07326

Put price:

4.07326

Call price from parity:

0.204058

Put-call parity holds

Batch 4

Call price:

92.1757

Put price from parity:

1.2475

Put price:

1.2475

Call price from parity:

92.1757

Put-call parity holds

A.1.c:

Varying underlying price:

1.10869e-13

1.09147e-11

5.13441e-10

1.336e-08

2.1465e-07

2.31709e-06

1.79531e-05

0.00010523

0.000486801

0.00183997

0.00584714

0.0160004

0.0384699

0.0826721

0.161146

0.288544

0.479857

0.748326

1.10357

1.55041

2.08855

2.71316

3.41591

4.18638

5.01333

5.88578

6.79376

7.72868

8.6835

9.65268

10.632

11.6183

12.6094

13.6037

14.6

15.5977

16.5963

17.5954

18.5949

19.5945

20.5943

A.1.d.i

Varying expiry time:

2.13337

5.6754

8.51879

11.0118

13.2724

15.3584

17.3034

A.1.d.ii

Varying volatility:

2.13337

2.7072

3.29033

3.87943

4.47244

5.06798

5.66509

A.1.d.iii

Varying strike price and risk-free rate:

14.9694

10.3417

6.41811

3.53371

1.72454

0.751569

0.295965

0.106711

A.2.a:

Delta

Call delta:

0.594629

Put delta:

-0.356601

Gamma:

0.0134936

A.2.b:

Call Delta (varying underlying price):

1.33568e-05

6.50166e-05

0.000245471

0.000756025

0.0019728

0.00449025

0.00911948

0.0168276

0.028625

0.0454245

0.0679055

0.0964112

0.130898

0.170938

0.215772

0.264393

0.315644

0.368319

0.421251

0.473376

0.523785

0.571745

0.616709

0.658306

0.696329

0.730708

0.761485

0.788791

0.812819

0.833806

0.852012

0.867708

0.881162

0.892636

0.902375

0.910605

0.917532

0.923342

0.928198

0.932245

0.935609

Put Delta (varying underlying price):

-0.951216

-0.951164

-0.950984

-0.950473

-0.949257

-0.946739

-0.94211

-0.934402

-0.922604

-0.905805

-0.883324

-0.854818

-0.820332

-0.780292

-0.735458

-0.686837

-0.635586

-0.58291

-0.529979

-0.477853

-0.427444

-0.379484

-0.334521

-0.292923

-0.2549

-0.220522

-0.189744

-0.162438

-0.13841

-0.117423

-0.0992172

-0.0835217

-0.070067

-0.058593

-0.0488544

-0.0406246

-0.0336973

-0.0278875

-0.0230312

-0.0189841

-0.0156208

Gamma (varying underlying price):

2.31163e-05

9.40062e-05

0.000299354

0.000783644

0.00174897

0.00342214

0.00599946

0.00958767

0.0141616

0.0195524

0.0254676

0.0315364

0.0373646

0.0425875

0.0469111

0.0501358

0.0521637

0.0529914

0.0526942

0.0514046

0.0492911

0.0465377

0.0433279

0.0398326

0.0362023

0.0325627

0.0290138

0.0256307

0.0224657

0.0195517

0.0169054

0.0145307

0.0124221

0.0105669

0.00894811

0.00754596

0.00633945

0.00530739

0.00442928

0.00368575

0.00305891

A.2.c:

Call Delta (varying expiry time and volatility):

0.594629

0.566667

0.551344

0.540009

0.530347

0.521366

0.512531

Put Delta (varying expiry time and volatility):

-0.356601

-0.33817

-0.309364

-0.278722

-0.248454

-0.219452

-0.192157

Gamma (varying expiry time and volatility):

0.0134936

0.00816056

0.00568773

0.00421089

0.00322193

0.00251459

0.00198667

A.2.d:

Call delta (divided differences with h = 1):

0.59458

Call delta (divided differences with h = 0.5):

0.594617

Call delta (divided differences with h = 0.1):

0.594628

Put delta (divided differences with h = 1):

-0.356649

Put delta (divided differences with h = 0.5):

-0.356613

Put delta (divided differences with h = 0.1):

-0.356601

Gamma (divided differences with h = 1):

0.0134928

Gamma (divided differences with h = 0.5):

0.0134934

Gamma (divided differences with h = 0.1):

0.0134936

Call delta (divided differences with h = 1):

1.97655e-05

8.58619e-05

0.000299505

0.000872484

0.00218833

0.00484131

0.00963238

0.0175097

0.02946

0.0463733

0.0689118

0.0974102

0.131825

0.171738

0.216404

0.264831

0.315881

0.368361

0.421116

0.473089

0.523376

0.571245

0.616147

0.65771

0.695721

0.730107

0.760906

0.788244

0.812312

0.833343

0.851594

0.867334

0.880832

0.892347

0.902124

0.910388

0.917346

0.923183

0.928063

0.932131

0.935513

Call delta (divided differences with h = 0.5):

1.48735e-05

7.00537e-05

0.000258703

0.000784788

0.00202633

0.00457776

0.00924763

0.0169983

0.0288342

0.0456623

0.0681577

0.0966616

0.13113

0.171138

0.21593

0.264502

0.315703

0.368329

0.421217

0.473304

0.523683

0.57162

0.616568

0.658157

0.696177

0.730558

0.76134

0.788654

0.812693

0.83369

0.851908

0.867614

0.88108

0.892564

0.902312

0.910551

0.917486

0.923302

0.928165

0.932217

0.935585

Call delta (divided differences with h = 0.1):

1.34164e-05

6.52159e-05

0.000245997

0.000757171

0.00197494

0.00449375

0.00912461

0.0168345

0.0286334

0.0454341

0.0679156

0.0964213

0.130907

0.170946

0.215778

0.264397

0.315646

0.368319

0.421249

0.473373

0.523781

0.57174

0.616703

0.6583

0.696323

0.730702

0.761479

0.788786

0.812814

0.833802

0.852008

0.867704

0.881159

0.892634

0.902372

0.910603

0.91753

0.92334

0.928197

0.932244

0.935608

Put delta (divided differences with h = 1):

-0.95121

-0.951144

-0.95093

-0.950357

-0.949041

-0.946388

-0.941597

-0.93372

-0.921769

-0.904856

-0.882318

-0.853819

-0.819404

-0.779491

-0.734826

-0.686398

-0.635349

-0.582868

-0.530114

-0.47814

-0.427853

-0.379985

-0.335083

-0.29352

-0.255508

-0.221122

-0.190323

-0.162985

-0.138917

-0.117887

-0.0996356

-0.0838951

-0.070397

-0.0588821

-0.0491058

-0.0408416

-0.0338834

-0.0280463

-0.023166

-0.019098

-0.0157166

Put delta (divided differences with h = 0.5):

-0.951215

-0.951159

-0.950971

-0.950445

-0.949203

-0.946652

-0.941982

-0.934231

-0.922395

-0.905567

-0.883072

-0.854568

-0.820099

-0.780091

-0.735299

-0.686727

-0.635527

-0.5829

-0.530013

-0.477925

-0.427547

-0.37961

-0.334661

-0.293073

-0.255052

-0.220672

-0.189889

-0.162575

-0.138537

-0.117539

-0.0993218

-0.083615

-0.0701495

-0.0586653

-0.0489172

-0.0406788

-0.0337438

-0.0279272

-0.0230649

-0.0190126

-0.0156448

Put delta (divided differences with h = 0.1):

-0.951216

-0.951164

-0.950983

-0.950472

-0.949254

-0.946736

-0.942105

-0.934395

-0.922596

-0.905795

-0.883314

-0.854808

-0.820322

-0.780284

-0.735451

-0.686832

-0.635583

-0.58291

-0.52998

-0.477856

-0.427448

-0.379489

-0.334526

-0.292929

-0.254906

-0.220528

-0.18975

-0.162444

-0.138415

-0.117428

-0.0992214

-0.0835254

-0.0700703

-0.0585959

-0.0488569

-0.0406267

-0.0336991

-0.0278891

-0.0230325

-0.0189853

-0.0156218

Gamma (divided differences with h = 1):

2.72512e-05

0.000104942

0.000322343

0.000823616

0.00180809

0.00349787

0.00608427

0.00967043

0.0142302

0.0195964

0.0254805

0.0315163

0.0373138

0.0425121

0.0468191

0.0500357

0.0520634

0.0528975

0.0526113

0.0513359

0.0492378

0.0464998

0.0433042

0.0398215

0.0362016

0.0325703

0.0290277

0.025649

0.0224867

0.0195741

0.0169281

0.0145529

0.0124432

0.0105866

0.00896617

0.00756227

0.00635398

0.00532022

0.00444049

0.00369546

0.00306727

Gamma (divided differences with h = 0.5):

2.41258e-05

9.67062e-05

0.000305069

0.000793622

0.00176377

0.00344112

0.00602073

0.00960843

0.0141788

0.0195634

0.0254709

0.0315314

0.0373518

0.0425686

0.046888

0.0501108

0.0521386

0.0529679

0.0526735

0.0513874

0.0492778

0.0465282

0.043322

0.0398298

0.0362021

0.0325646

0.0290173

0.0256353

0.022471

0.0195573

0.0169111

0.0145363

0.0124274

0.0105718

0.00895262

0.00755004

0.00634308

0.0053106

0.00443208

0.00368817

0.003061

Gamma (divided differences with h = 0.1):

2.31564e-05

9.41138e-05

0.000299582

0.000784043

0.00174957

0.0034229

0.00600031

0.0095885

0.0141623

0.0195528

0.0254678

0.0315362

0.0373641

0.0425867

0.0469102

0.0501348

0.0521627

0.0529905

0.0526933

0.0514039

0.0492906

0.0465374

0.0433277

0.0398325

0.0362022

0.0325627

0.0290139

0.0256309

0.0224659

0.0195519

0.0169056

0.014531

0.0124223

0.0105671

0.00894829

0.00754613

0.00633959

0.00530752

0.00442939

0.00368584

0.00305899

Comparing accuracy with different h (with part A.2.a):

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | h=1 | |  | h=0.5 | |  | h=0.1 | |
|  | Exact | Estimate | Deviation |  | Estimate | Deviation |  | Estimate | Deviation |
| Call delta | 0.594629 | 0.59458 | -0.0082% |  | 0.594617 | -0.0020% |  | 0.594628 | -0.0002% |
| Put delta | -0.3566 | -0.35665 | 0.0135% |  | -0.35661 | 0.0034% |  | -0.3566 | 0.0000% |
| Gamma | 0.013494 | 0.013493 | -0.0059% |  | 0.013493 | -0.0015% |  | 0.013494 | 0.0000% |

Comparing accuracy with different h (with part A.2.b):

Call delta:

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | h=1 | |  | h=0.5 | |  | h=0.1 | |
| Exact | Estimate | Deviation |  | Estimate | Deviation |  | Estimate | Deviation |
| 1.34E-05 | 1.98E-05 | 47.9808% |  | 1.49E-05 | 11.3553% |  | 1.34E-05 | 0.4462% |
| 6.50E-05 | 8.59E-05 | 32.0615% |  | 7.01E-05 | 7.7474% |  | 6.52E-05 | 0.3065% |
| 0.000245 | 0.0003 | 22.0124% |  | 0.000259 | 5.3905% |  | 0.000246 | 0.2143% |
| 0.000756 | 0.000872 | 15.4041% |  | 0.000785 | 3.8045% |  | 0.000757 | 0.1516% |
| 0.001973 | 0.002188 | 10.9251% |  | 0.002026 | 2.7134% |  | 0.001975 | 0.1085% |
| 0.00449 | 0.004841 | 7.8183% |  | 0.004578 | 1.9489% |  | 0.004494 | 0.0779% |
| 0.009119 | 0.009632 | 5.6242% |  | 0.009248 | 1.4052% |  | 0.009125 | 0.0563% |
| 0.016828 | 0.01751 | 4.0535% |  | 0.016998 | 1.0144% |  | 0.016835 | 0.0410% |
| 0.028625 | 0.02946 | 2.9170% |  | 0.028834 | 0.7308% |  | 0.028633 | 0.0293% |
| 0.045425 | 0.046373 | 2.0887% |  | 0.045662 | 0.5235% |  | 0.045434 | 0.0211% |
| 0.067906 | 0.068912 | 1.4819% |  | 0.068158 | 0.3714% |  | 0.067916 | 0.0149% |
| 0.096411 | 0.09741 | 1.0362% |  | 0.096662 | 0.2597% |  | 0.096421 | 0.0105% |
| 0.130898 | 0.131825 | 0.7082% |  | 0.13113 | 0.1772% |  | 0.130907 | 0.0069% |
| 0.170938 | 0.171738 | 0.4680% |  | 0.171138 | 0.1170% |  | 0.170946 | 0.0047% |
| 0.215772 | 0.216404 | 0.2929% |  | 0.21593 | 0.0732% |  | 0.215778 | 0.0028% |
| 0.264393 | 0.264831 | 0.1657% |  | 0.264502 | 0.0412% |  | 0.264397 | 0.0015% |
| 0.315644 | 0.315881 | 0.0751% |  | 0.315703 | 0.0187% |  | 0.315646 | 0.0006% |
| 0.368319 | 0.368361 | 0.0114% |  | 0.368329 | 0.0027% |  | 0.368319 | 0.0000% |
| 0.421251 | 0.421116 | -0.0320% |  | 0.421217 | -0.0081% |  | 0.421249 | -0.0005% |
| 0.473376 | 0.473089 | -0.0606% |  | 0.473304 | -0.0152% |  | 0.473373 | -0.0006% |
| 0.523785 | 0.523376 | -0.0781% |  | 0.523683 | -0.0195% |  | 0.523781 | -0.0008% |
| 0.571745 | 0.571245 | -0.0875% |  | 0.57162 | -0.0219% |  | 0.57174 | -0.0009% |
| 0.616709 | 0.616147 | -0.0911% |  | 0.616568 | -0.0229% |  | 0.616703 | -0.0010% |
| 0.658306 | 0.65771 | -0.0905% |  | 0.658157 | -0.0226% |  | 0.6583 | -0.0009% |
| 0.696329 | 0.695721 | -0.0873% |  | 0.696177 | -0.0218% |  | 0.696323 | -0.0009% |
| 0.730708 | 0.730107 | -0.0822% |  | 0.730558 | -0.0205% |  | 0.730702 | -0.0008% |
| 0.761485 | 0.760906 | -0.0760% |  | 0.76134 | -0.0190% |  | 0.761479 | -0.0008% |
| 0.788791 | 0.788244 | -0.0693% |  | 0.788654 | -0.0174% |  | 0.788786 | -0.0006% |
| 0.812819 | 0.812312 | -0.0624% |  | 0.812693 | -0.0155% |  | 0.812814 | -0.0006% |
| 0.833806 | 0.833343 | -0.0555% |  | 0.83369 | -0.0139% |  | 0.833802 | -0.0005% |
| 0.852012 | 0.851594 | -0.0491% |  | 0.851908 | -0.0122% |  | 0.852008 | -0.0005% |
| 0.867708 | 0.867334 | -0.0431% |  | 0.867614 | -0.0108% |  | 0.867704 | -0.0005% |
| 0.881162 | 0.880832 | -0.0375% |  | 0.88108 | -0.0093% |  | 0.881159 | -0.0003% |
| 0.892636 | 0.892347 | -0.0324% |  | 0.892564 | -0.0081% |  | 0.892634 | -0.0002% |
| 0.902375 | 0.902124 | -0.0278% |  | 0.902312 | -0.0070% |  | 0.902372 | -0.0003% |
| 0.910605 | 0.910388 | -0.0238% |  | 0.910551 | -0.0059% |  | 0.910603 | -0.0002% |
| 0.917532 | 0.917346 | -0.0203% |  | 0.917486 | -0.0050% |  | 0.91753 | -0.0002% |
| 0.923342 | 0.923183 | -0.0172% |  | 0.923302 | -0.0043% |  | 0.92334 | -0.0002% |
| 0.928198 | 0.928063 | -0.0145% |  | 0.928165 | -0.0036% |  | 0.928197 | -0.0001% |
| 0.932245 | 0.932131 | -0.0122% |  | 0.932217 | -0.0030% |  | 0.932244 | -0.0001% |
| 0.935609 | 0.935513 | -0.0103% |  | 0.935585 | -0.0026% |  | 0.935608 | -0.0001% |
| Average |  | 3.76% |  |  | 0.91% |  |  | 0.04% |

Put delta:

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | h=1 | |  | h=0.5 | |  | h=0.1 | |
| Exact | Estimate | Deviation |  | Estimate | Deviation |  | Estimate | Deviation |
| -0.95122 | -0.95121 | -0.0006% |  | -0.95122 | -0.0001% |  | -0.95122 | 0.0000% |
| -0.95116 | -0.95114 | -0.0021% |  | -0.95116 | -0.0005% |  | -0.95116 | 0.0000% |
| -0.95098 | -0.95093 | -0.0057% |  | -0.95097 | -0.0014% |  | -0.95098 | -0.0001% |
| -0.95047 | -0.95036 | -0.0122% |  | -0.95045 | -0.0029% |  | -0.95047 | -0.0001% |
| -0.94926 | -0.94904 | -0.0228% |  | -0.9492 | -0.0057% |  | -0.94925 | -0.0003% |
| -0.94674 | -0.94639 | -0.0371% |  | -0.94665 | -0.0092% |  | -0.94674 | -0.0003% |
| -0.94211 | -0.9416 | -0.0545% |  | -0.94198 | -0.0136% |  | -0.94211 | -0.0005% |
| -0.9344 | -0.93372 | -0.0730% |  | -0.93423 | -0.0183% |  | -0.9344 | -0.0007% |
| -0.9226 | -0.92177 | -0.0905% |  | -0.9224 | -0.0227% |  | -0.9226 | -0.0009% |
| -0.90581 | -0.90486 | -0.1048% |  | -0.90557 | -0.0263% |  | -0.9058 | -0.0011% |
| -0.88332 | -0.88232 | -0.1139% |  | -0.88307 | -0.0285% |  | -0.88331 | -0.0011% |
| -0.85482 | -0.85382 | -0.1169% |  | -0.85457 | -0.0292% |  | -0.85481 | -0.0012% |
| -0.82033 | -0.8194 | -0.1131% |  | -0.8201 | -0.0284% |  | -0.82032 | -0.0012% |
| -0.78029 | -0.77949 | -0.1027% |  | -0.78009 | -0.0258% |  | -0.78028 | -0.0010% |
| -0.73546 | -0.73483 | -0.0859% |  | -0.7353 | -0.0216% |  | -0.73545 | -0.0010% |
| -0.68684 | -0.6864 | -0.0639% |  | -0.68673 | -0.0160% |  | -0.68683 | -0.0007% |
| -0.63559 | -0.63535 | -0.0373% |  | -0.63553 | -0.0093% |  | -0.63558 | -0.0005% |
| -0.58291 | -0.58287 | -0.0072% |  | -0.5829 | -0.0017% |  | -0.58291 | 0.0000% |
| -0.52998 | -0.53011 | 0.0255% |  | -0.53001 | 0.0064% |  | -0.52998 | 0.0002% |
| -0.47785 | -0.47814 | 0.0601% |  | -0.47793 | 0.0151% |  | -0.47786 | 0.0006% |
| -0.42744 | -0.42785 | 0.0957% |  | -0.42755 | 0.0241% |  | -0.42745 | 0.0009% |
| -0.37948 | -0.37999 | 0.1320% |  | -0.37961 | 0.0332% |  | -0.37949 | 0.0013% |
| -0.33452 | -0.33508 | 0.1680% |  | -0.33466 | 0.0419% |  | -0.33453 | 0.0015% |
| -0.29292 | -0.29352 | 0.2038% |  | -0.29307 | 0.0512% |  | -0.29293 | 0.0020% |
| -0.2549 | -0.25551 | 0.2385% |  | -0.25505 | 0.0596% |  | -0.25491 | 0.0024% |
| -0.22052 | -0.22112 | 0.2721% |  | -0.22067 | 0.0680% |  | -0.22053 | 0.0027% |
| -0.18974 | -0.19032 | 0.3051% |  | -0.18989 | 0.0764% |  | -0.18975 | 0.0032% |
| -0.16244 | -0.16299 | 0.3367% |  | -0.16258 | 0.0843% |  | -0.16244 | 0.0037% |
| -0.13841 | -0.13892 | 0.3663% |  | -0.13854 | 0.0918% |  | -0.13842 | 0.0036% |
| -0.11742 | -0.11789 | 0.3952% |  | -0.11754 | 0.0988% |  | -0.11743 | 0.0043% |
| -0.09922 | -0.09964 | 0.4217% |  | -0.09932 | 0.1054% |  | -0.09922 | 0.0042% |
| -0.08352 | -0.0839 | 0.4471% |  | -0.08362 | 0.1117% |  | -0.08353 | 0.0044% |
| -0.07007 | -0.0704 | 0.4710% |  | -0.07015 | 0.1177% |  | -0.07007 | 0.0047% |
| -0.05859 | -0.05888 | 0.4934% |  | -0.05867 | 0.1234% |  | -0.0586 | 0.0049% |
| -0.04885 | -0.04911 | 0.5146% |  | -0.04892 | 0.1285% |  | -0.04886 | 0.0051% |
| -0.04062 | -0.04084 | 0.5342% |  | -0.04068 | 0.1334% |  | -0.04063 | 0.0052% |
| -0.0337 | -0.03388 | 0.5523% |  | -0.03374 | 0.1380% |  | -0.0337 | 0.0053% |
| -0.02789 | -0.02805 | 0.5694% |  | -0.02793 | 0.1424% |  | -0.02789 | 0.0057% |
| -0.02303 | -0.02317 | 0.5853% |  | -0.02306 | 0.1463% |  | -0.02303 | 0.0056% |
| -0.01898 | -0.0191 | 0.6000% |  | -0.01901 | 0.1501% |  | -0.01899 | 0.0063% |
| -0.01562 | -0.01572 | 0.6133% |  | -0.01564 | 0.1536% |  | -0.01562 | 0.0064% |
| Average |  | 0.18% |  |  | 0.04% |  |  | 0.00% |

Gamma:

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | h=1 | |  | h=0.5 | |  | h=0.1 | |
| Exact | Estimate | Deviation |  | Estimate | Deviation |  | Estimate | Deviation |
| 2.31E-05 | 2.73E-05 | 17.8874% |  | 2.41E-05 | 4.3670% |  | 2.32E-05 | 0.1735% |
| 9.40E-05 | 0.000105 | 11.6331% |  | 9.67E-05 | 2.8722% |  | 9.41E-05 | 0.1145% |
| 0.000299 | 0.000322 | 7.6795% |  | 0.000305 | 1.9091% |  | 0.0003 | 0.0762% |
| 0.000784 | 0.000824 | 5.1008% |  | 0.000794 | 1.2733% |  | 0.000784 | 0.0509% |
| 0.001749 | 0.001808 | 3.3803% |  | 0.001764 | 0.8462% |  | 0.00175 | 0.0343% |
| 0.003422 | 0.003498 | 2.2129% |  | 0.003441 | 0.5546% |  | 0.003423 | 0.0222% |
| 0.005999 | 0.006084 | 1.4136% |  | 0.006021 | 0.3545% |  | 0.006 | 0.0142% |
| 0.009588 | 0.00967 | 0.8632% |  | 0.009608 | 0.2165% |  | 0.009589 | 0.0087% |
| 0.014162 | 0.01423 | 0.4844% |  | 0.014179 | 0.1215% |  | 0.014162 | 0.0049% |
| 0.019552 | 0.019596 | 0.2250% |  | 0.019563 | 0.0563% |  | 0.019553 | 0.0020% |
| 0.025468 | 0.025481 | 0.0507% |  | 0.025471 | 0.0130% |  | 0.025468 | 0.0008% |
| 0.031536 | 0.031516 | -0.0637% |  | 0.031531 | -0.0159% |  | 0.031536 | -0.0006% |
| 0.037365 | 0.037314 | -0.1360% |  | 0.037352 | -0.0343% |  | 0.037364 | -0.0013% |
| 0.042588 | 0.042512 | -0.1770% |  | 0.042569 | -0.0444% |  | 0.042587 | -0.0019% |
| 0.046911 | 0.046819 | -0.1961% |  | 0.046888 | -0.0492% |  | 0.04691 | -0.0019% |
| 0.050136 | 0.050036 | -0.1997% |  | 0.050111 | -0.0499% |  | 0.050135 | -0.0020% |
| 0.052164 | 0.052063 | -0.1923% |  | 0.052139 | -0.0481% |  | 0.052163 | -0.0019% |
| 0.052991 | 0.052898 | -0.1772% |  | 0.052968 | -0.0443% |  | 0.052991 | -0.0017% |
| 0.052694 | 0.052611 | -0.1573% |  | 0.052674 | -0.0393% |  | 0.052693 | -0.0017% |
| 0.051405 | 0.051336 | -0.1336% |  | 0.051387 | -0.0335% |  | 0.051404 | -0.0014% |
| 0.049291 | 0.049238 | -0.1081% |  | 0.049278 | -0.0270% |  | 0.049291 | -0.0010% |
| 0.046538 | 0.0465 | -0.0814% |  | 0.046528 | -0.0204% |  | 0.046537 | -0.0006% |
| 0.043328 | 0.043304 | -0.0547% |  | 0.043322 | -0.0136% |  | 0.043328 | -0.0005% |
| 0.039833 | 0.039822 | -0.0279% |  | 0.03983 | -0.0070% |  | 0.039833 | -0.0003% |
| 0.036202 | 0.036202 | -0.0019% |  | 0.036202 | -0.0006% |  | 0.036202 | -0.0003% |
| 0.032563 | 0.03257 | 0.0233% |  | 0.032565 | 0.0058% |  | 0.032563 | 0.0000% |
| 0.029014 | 0.029028 | 0.0479% |  | 0.029017 | 0.0121% |  | 0.029014 | 0.0003% |
| 0.025631 | 0.025649 | 0.0714% |  | 0.025635 | 0.0179% |  | 0.025631 | 0.0008% |
| 0.022466 | 0.022487 | 0.0935% |  | 0.022471 | 0.0236% |  | 0.022466 | 0.0009% |
| 0.019552 | 0.019574 | 0.1146% |  | 0.019557 | 0.0286% |  | 0.019552 | 0.0010% |
| 0.016905 | 0.016928 | 0.1343% |  | 0.016911 | 0.0337% |  | 0.016906 | 0.0012% |
| 0.014531 | 0.014553 | 0.1528% |  | 0.014536 | 0.0385% |  | 0.014531 | 0.0021% |
| 0.012422 | 0.012443 | 0.1699% |  | 0.012427 | 0.0427% |  | 0.012422 | 0.0016% |
| 0.010567 | 0.010587 | 0.1864% |  | 0.010572 | 0.0464% |  | 0.010567 | 0.0019% |
| 0.008948 | 0.008966 | 0.2018% |  | 0.008953 | 0.0504% |  | 0.008948 | 0.0020% |
| 0.007546 | 0.007562 | 0.2161% |  | 0.00755 | 0.0541% |  | 0.007546 | 0.0023% |
| 0.006339 | 0.006354 | 0.2292% |  | 0.006343 | 0.0573% |  | 0.00634 | 0.0022% |
| 0.005307 | 0.00532 | 0.2417% |  | 0.005311 | 0.0605% |  | 0.005308 | 0.0024% |
| 0.004429 | 0.00444 | 0.2531% |  | 0.004432 | 0.0632% |  | 0.004429 | 0.0025% |
| 0.003686 | 0.003695 | 0.2634% |  | 0.003688 | 0.0657% |  | 0.003686 | 0.0024% |
| 0.003059 | 0.003067 | 0.2733% |  | 0.003061 | 0.0683% |  | 0.003059 | 0.0026% |
| Average |  | 1.27% |  |  | 0.31% |  |  | 0.01% |

As expected, accuracy of approximation increases as h get smaller.